

# Differential-Operator Representations of $S_n$ and Singular Vectors in Verma Modules <sup>1</sup>

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## Abstract

Given a weight of  $sl(n, \mathbb{C})$ , we derive a system of variable-coefficient second-order linear partial differential equations that determines the singular vectors in the corresponding Verma module, and a differential-operator representation of the symmetric group  $S_n$  on the related space of truncated power series. We prove that the solution space of the system of partial differential equations is exactly spanned by  $\{\sigma(1) \mid \sigma \in S_n\}$ . Moreover, the singular vectors of  $sl(n, \mathbb{C})$  in the Verma module are given by those  $\sigma(1)$  that are polynomials. The well-known results of Verma, Bernstein-Gel'fand-Gel'fand and Jantzen for the case of  $sl(n, \mathbb{C})$  are naturally included in our almost elementary approach of partial differential equations.

## 1 Introduction

One of the most beautiful things in Lie algebras is the highest weight representation theory. It was established based on the induced modules of a Lie algebra with respect to a Cartan decomposition from one-dimensional modules of the Borel subalgebra associated with a linear function (weight) on the Cartan subalgebra. These modules are now known as *Verma modules* [V1]. A *singular vector* (or *canonical vector*) in a Verma module is a weight vector annihilated by positive root vectors. It is well known that the structure of a Verma module of a finite-dimensional simple Lie algebra is completely determined by its singular vectors (cf. [V1]). In this paper, we find explicit formulas for singular vectors in Verma modules for the Lie algebra  $sl(n, \mathbb{C})$  in terms of a differential-operator representation of the symmetric group  $S_n$  on a certain space of truncated power series.

The structure of Verma module was first studied by Verma [V1]. Verma reduced the problem of determining all submodules of a Verma module of a finite-dimensional semisimple Lie algebra to determining the embeddings of the other Verma modules into

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the objective module. He proved that the multiplicity of the embedding is at most one. Bernstein, Gel'fand and Gel'fand [BGG] introduced the well-known useful notion of category  $\mathcal{O}$  of representations, and found a necessary and sufficient condition for the existence of such a embedding in terms of the action of Weyl group on weights. Sapovolov [S] introduced a certain bilinear form on a universal enveloping algebra. Lepowsky [L1-L4] studied analogous induced modules with respect to Iwasawa decomposition that is more general than Cartan decomposition, and obtained similar results as those in [V1] and [BGG]. These modules are now known as *generalized Verma modules*.

Jantzen [J1, J2] introduced his famous “Jantzen filtrations” on Verma modules and used Sapovolov form to determine weights of singular vectors in Verma modules. Verma modules of infinite-dimensional Lie algebras were first studied by Kac [Kv1]. Kac and Kazhdan [KK] generalized the results of Verma [V1] and Bernstein-Gel'fand-Gel'fand [BGG] to the contragredient Lie algebra corresponding a symmetrizable generalized Cartan matrix. Deodhar, Gabber and Kac [DGK] generalized the results further to more general matrix Lie algebras. Rocha-Caridi and Wallach [RW1, RW2] generalized the results of Verma [V1] and Bernstein-Gel'fand-Gel'fand [BGG] to a class of graded Lie algebras possessing a Cartan decomposition and obtained Jantzen's character formula corresponding to the quotient of two Verma modules. The resolutions of irreducible highest weight modules over rank-2 Kac-Moody algebras were constructed.

One of the fundamental and difficult remaining problems in this direction is how to determine the singular vectors explicitly. Malikov, Feigin and Fuchs [MFF] introduced a formal manipulation on products of several general powers of negative simple root vectors and used free Lie algebras to give a rough condition when such product is well defined. It seems to us that their condition can not be verified in general and their method can practically be applied only to finding very special singular vectors.

In this paper, we introduce an almost elementary partial differential equation approach of determining the singular vectors in any Verma module of  $sl(n, \mathbb{C})$ . First, we identify the Verma modules with a space of polynomials, and the action of  $sl(n, \mathbb{C})$  on the Verma module is identified with a differential operator action of  $sl(n, \mathbb{C})$  on the polynomials. Any singular vector in the Verma module becomes a polynomial solution of a system of variable-coefficient second-order linear partial differential equations. Thus we have changed a difficult problem in a noncommutative space to a problem in commutative space. However, it is in general impossible to solve the system in the space of polynomials. So we extend the action of  $sl(n, \mathbb{C})$  on the polynomial space to a larger space of certain truncated formal power series. On this larger space, the negative simple root vectors become differential operators whose arbitrary complex powers are well defined (so are

their products). In this way, we overcome the difficulty of determining whether a product of several general powers of negative simple root vectors is well defined in the work [MFF] of Malikov, Feigin and Fuchs. Next we define a differential-operator representation of the symmetric group  $S_n$  on the space of truncated power series. Using commutator relations among root vectors and a certain substitution-of-variable technique that we developed in [X], we prove that the solution space of the system of partial differential equations in the space of truncated power series is exactly spanned by  $\{\sigma(1) \mid \sigma \in S_n\}$ . Moreover, the singular vectors of  $sl(n, \mathbb{C})$  in the Verma module are given by those  $\sigma(1)$  that are polynomials. In particular, there are exactly  $n!$  singular vectors up to scalar multiples in the Verma module when the weight is dominant integral.

In Section 2, we derive the system of partial differential equations and a differential-operator representation of the symmetric group  $S_n$  on the space of certain truncated formal power series. Moreover, we prove that  $\{\sigma(1) \mid \sigma \in S_n\}$  are the solutions of the system. In Section 3, we completely solve the system in the space of power series.

## 2 Differential Equations and Representations

In this section, we first derive a system of variable-coefficient second-order partial differential equations that determines the singular vectors in the Verma modules over the special linear Lie algebra  $sl(n, \mathbb{C})$ . Then we construct a differential operator representation of the symmetric group  $S_n$  on the related space of truncated formal power series and prove that  $\{\sigma(1) \mid \sigma \in S_n\}$  are the solutions of the system in the space.

Denote by  $E_{i,j}$  the square matrix with 1 as its  $(i, j)$ -entry and 0 as the others. The special linear Lie algebra

$$sl(n, \mathbb{C}) = \sum_{1 \leq i < j \leq n} (\mathbb{C}E_{i,j} + \mathbb{C}E_{j,i}) + \sum_{r=1}^{n-1} \mathbb{C}(E_{r,r} - E_{r+1,r+1}) \quad (2.1)$$

with the Lie bracket:

$$[A, B] = AB - BA \quad \text{for } A, B \in sl(n, \mathbb{C}). \quad (2.2)$$

Set

$$h_i = E_{i,i} - E_{i+1,i+1}, \quad i = 1, 2, \dots, n-1. \quad (2.3)$$

The subspace

$$H = \sum_{i=1}^{n-1} \mathbb{C}h_i \quad (2.4)$$

forms a Cartan subalgebra of  $sl(n, \mathbb{C})$ . We choose

$$\{E_{i,j} \mid 1 \leq i < j \leq n\} \text{ as positive root vectors.} \quad (2.5)$$

In particular, we have

$$\{E_{i,i+1} \mid i = 1, 2, \dots, n-1\} \text{ as positive simple root vectors.} \quad (2.6)$$

Accordingly,

$$\{E_{i,j} \mid 1 \leq j < i \leq n\} \text{ are negative root vectors} \quad (2.7)$$

and we have

$$\{E_{i+1,i} \mid i = 1, 2, \dots, n-1\} \text{ as negative simple root vectors.} \quad (2.8)$$

Denote by  $\mathbb{N}$  the additive semigroup of nonnegative integers. Let

$$\Gamma = \sum_{1 \leq j < i \leq n} \mathbb{N} \epsilon_{i,j} \quad (2.9)$$

be the torsion-free additive semigroup of rank  $n(n-1)/2$  with  $\epsilon_{i,j}$  as base elements. Let  $\mathcal{G}_-$  be the Lie subalgebra spanned by (2.7) and let  $U(\mathcal{G}_-)$  be its universal enveloping algebra. For

$$\alpha = \sum_{1 \leq j < i \leq n} \alpha_{i,j} \epsilon_{i,j} \in \Gamma, \quad (2.10)$$

we denote

$$E^\alpha = E_{2,1}^{\alpha_{2,1}} E_{3,1}^{\alpha_{3,1}} E_{3,2}^{\alpha_{3,2}} E_{4,1}^{\alpha_{4,1}} \cdots E_{n,1}^{\alpha_{n,1}} \cdots E_{n,n-1}^{\alpha_{n,n-1}} \in U(\mathcal{G}_-). \quad (2.11)$$

Then

$$\{E^\alpha \mid \alpha \in \Gamma\} \text{ forms a basis of } U(\mathcal{G}_-). \quad (2.12)$$

Let  $\lambda$  be a weight, which is a linear function on  $H$ , such that

$$\lambda(h_i) = \lambda_i \quad \text{for } i = 1, 2, \dots, n-1. \quad (2.13)$$

Recall that  $sl(n, \mathbb{C})$  is generated by  $\{E_{i,i+1}, E_{i+1,i} \mid i = 1, 2, \dots, n-1\}$  as a Lie algebra. The Verma  $sl(n, \mathbb{C})$ -module with the highest-weight vector  $v_\lambda$  of weight  $\lambda$  is given by

$$M_\lambda = \text{Span}\{E^\alpha v_\lambda \mid \alpha \in \Gamma\}, \quad (2.14)$$

with the action determined by

$$\begin{aligned} E_{i,i+1}(E^\alpha v_\lambda) &= \left( \sum_{j=1}^{i-1} \alpha_{i+1,j} E^{\alpha + \epsilon_{i,j} - \epsilon_{i+1,j}} - \sum_{j=i+2}^n \alpha_{j,i} E^{\alpha + \epsilon_{j,i+1} - \epsilon_{j,i}} \right. \\ &\quad \left. + \alpha_{i+1,i}(\lambda_i + 1 - \sum_{j=i+1}^n \alpha_{j,i} + \sum_{j=i+2}^n \alpha_{j,i+1}) E^{\alpha - \epsilon_{i+1,i}} \right) v_\lambda, \end{aligned} \quad (2.15)$$

$$E_{i+1,i}(E^\alpha v_\lambda) = (E^{\alpha + \epsilon_{i+1,i}} + \sum_{j=1}^{i-1} \alpha_{i,j} E^{\alpha + \epsilon_{i+1,j} - \epsilon_{i,j}}) v_\lambda \quad (2.16)$$

for  $i = 1, \dots, n-1$ . For any  $\alpha \in \Gamma$ , we define the *weight* of  $E^\alpha v_\lambda$  by

$$(\text{wt } E^\alpha v_\lambda)(h_i) = (\lambda_i + \sum_{p=1}^{i-1} (\alpha_{i,p} - \alpha_{i+1,p}) + \sum_{j=i+2}^n (\alpha_{j,i+1} - \alpha_{j,i}) - 2\alpha_{i+1,i})h_i \quad (2.17)$$

for  $i = 1, \dots, n-1$ . Then the Verma module  $M_\lambda$  is a space graded by weights. A *singular vector* is a homogeneous nonzero vector  $u$  in  $M_\lambda$  such that

$$E_{i,i+1}(u) = 0 \quad \text{for } i = 1, \dots, n-1. \quad (2.18)$$

Here we have used the fact that all positive root vectors are generated by simple positive root vectors. The Verma module is irreducible if and only if any singular vector is a scalar multiple of  $v_\lambda$ .

Consider the polynomial algebra

$$\mathcal{A} = \mathbb{C}[x_{i,j} \mid 1 \leq j < i \leq n] \quad (2.19)$$

in  $n(n-1)/2$  variables. Set

$$x^\alpha = \prod_{1 \leq j < i \leq n} x_{i,j}^{\alpha_{i,j}} \quad \text{for } \alpha \in \Gamma. \quad (2.20)$$

Then

$$\{x^\alpha \mid \alpha \in \Gamma\} \text{ forms a basis of } \mathcal{A}. \quad (2.21)$$

Thus we have a linear isomorphism  $\tau : M_\lambda \rightarrow \mathcal{A}$  determined by

$$\tau(E^\alpha v_\lambda) = x^\alpha \quad \text{for } \alpha \in \Gamma. \quad (2.22)$$

The algebra  $\mathcal{A}$  becomes  $sl(n, \mathbb{C})$ -module by the action

$$A(f) = \tau(A(\tau^{-1}(f))) \quad \text{for } A \in sl(n, \mathbb{C}), f \in \mathcal{A}. \quad (2.23)$$

For convenience, we denote the partial derivatives

$$\partial_{i,j} = \partial_{x_{i,j}} \quad \text{for } 1 \leq j < i \leq n. \quad (2.24)$$

In particular,

$$\begin{aligned} d_i &= E_{i,i+1}|_{\mathcal{A}} \\ &= (\lambda_i - \sum_{j=i+1}^n x_{j,i} \partial_{j,i} + \sum_{j=i+2}^n x_{j,i+1} \partial_{j,i+1}) \partial_{i+1,i} + \sum_{j=1}^{i-1} x_{i,j} \partial_{i+1,j} - \sum_{j=i+2}^n x_{j,i+1} \partial_{j,i} \end{aligned} \quad (2.25)$$

for  $i = 1, 2, \dots, n-1$  by (2.15).

**Proposition 2.1.** *A homogeneous vector  $u \in M_\lambda$  is a singular vector if and only if*

$$d_i(\tau(u)) = 0 \quad \text{for } i = 1, 2, \dots, n-1. \quad (2.26)$$

The system of partial differential equations

$$\begin{aligned} & (\lambda_i - \sum_{j=i+1}^n x_{j,i} \partial_{j,i} + \sum_{j=i+2}^n x_{j,i+1} \partial_{j,i+1}) \partial_{i+1,i}(z) \\ & + \sum_{j=1}^{i-1} x_{i,j} \partial_{i+1,j}(z) - \sum_{j=i+2}^n x_{j,i+1} \partial_{j,i}(z) = 0 \end{aligned} \quad (2.27)$$

for  $i = 1, 2, \dots, n-1$  and unknown function  $z$  in  $\{x_{i,j} \mid 1 \leq j < i \leq n\}$ , is called the *system of partial differential equations for the singular vectors of  $sl(n, \mathbb{C})$* .

Next we want to construct a differential operator representation of the symmetric group  $S_n$  on the related space of truncated series and prove that  $\{\sigma(1) \mid \sigma \in S_n\}$  are the solutions of the system. First, we have

$$\eta_i = E_{i+1,i}|_{\mathcal{A}} = x_{i+1,i} + \sum_{j=1}^{i-1} x_{i+1,j} \partial_{i,j} \quad (2.28)$$

for  $i = 1, 2, \dots, n-1$  by (2.16). Now we view  $\{d_i, \eta_i \mid i = 1, 2, \dots, n-1\}$  purely as differential operators acting on functions of  $\{x_{i,j} \mid 1 \leq j < i \leq n\}$ . In this way, we get a Lie algebra action on functions of  $\{x_{i,j} \mid 1 \leq j < i \leq n\}$  through  $E_{i,i+1} = d_i$  and  $E_{i+1,i} = \eta_i$  because  $sl(n, \mathbb{C})$  is generated by  $\{E_{i,i+1}, E_{i+1,i} \mid i = 1, 2, \dots, n-1\}$  as a Lie algebra. Note that

$$h_i(E^\alpha v_\lambda) = (\lambda_i + \sum_{p=1}^{i-1} (\alpha_{i,p} - \alpha_{i+1,p}) + \sum_{j=i+2}^n (\alpha_{j,i+1} - \alpha_{j,i}) - 2\alpha_{i+1,i}) E^\alpha v_\lambda \quad (2.29)$$

for  $i = 1, 2, \dots, n-1$  and  $\alpha \in \Gamma$ . Accordingly, we set

$$\zeta_i = h_i|_{\mathcal{A}} = \lambda_i + \sum_{p=1}^{i-1} (x_{i,p} \partial_{i,p} - x_{i+1,p} \partial_{i+1,p}) + \sum_{j=i+2}^n (x_{j,i+1} \partial_{j,i+1} - x_{j,i} \partial_{j,i}) - 2x_{i+1,i} \partial_{i+1,i} \quad (2.30)$$

for  $i = 1, 2, \dots, n-1$ . The elements  $h_i$  act on functions of  $\{x_{i,j} \mid 1 \leq j < i \leq n\}$  through  $\zeta_i$ . A function  $f$  of  $\{x_{i,j} \mid 1 \leq j < i \leq n\}$  is called *weighted* if there exist constants  $\mu_1, \mu_2, \dots, \mu_{n-1}$  such that

$$\zeta_i(f) = \mu_i f \quad \text{for } i = 1, 2, \dots, n-1. \quad (2.31)$$

Since  $d_i$  maps weighted functions to weighted functions, the system (2.27) is a weighted system. Any nonzero weighted solution of the system (2.27) is a singular vector of  $sl(n, \mathbb{C})$ .

In particular, any nonzero weighted polynomial solution  $f$  of the system (2.27) gives a singular vector  $\tau^{-1}(f)$  in the Verma module  $M_\lambda$ .

Let

$$\mathcal{A}_0 = \mathbb{C}[x_{i,j} \mid 1 \leq j \leq i-2 \leq n-2] \quad (2.32)$$

be the polynomial algebra in  $\{x_{i,j} \mid 1 \leq j \leq i-2 \leq n-2\}$ . We denote

$$x^{\vec{a}} = \prod_{i=1}^{n-1} x_{i+1,i}^{a_i} \quad \text{for } \vec{a} = (a_1, a_2, \dots, a_{n-1}) \in \mathbb{C}^{n-1}. \quad (2.33)$$

Let

$$\mathcal{A}_1 = \left\{ \sum_{\vec{j} \in \mathbb{N}^{n-1}} \sum_{i=1}^p f_{\vec{a}^i - \vec{j}} x^{\vec{a}^i - \vec{j}} \mid 1 \leq p \in \mathbb{N}, \vec{a}^i \in \mathbb{C}^{n-1}, f_{\vec{a}^i - \vec{j}} \in \mathcal{A}_0 \right\} \quad (2.34)$$

be the space of truncated-up formal power series in  $\{x_{2,1}, x_{3,2}, \dots, x_{n,n-1}\}$  over  $\mathcal{A}_0$ . Then  $\mathcal{A}$  is a subspace of  $\mathcal{A}_1$ . Since  $\mathcal{A}_1$  is invariant under the action of  $\{E_{i,i+1}|_{\mathcal{A}_1} = d_i, E_{i+1,i}|_{\mathcal{A}_1} = \eta_i \mid i = 1, 2, \dots, n-1\}$ ,  $\mathcal{A}_1$  becomes an  $sl(n, \mathbb{C})$ -module.

For  $a \in \mathbb{C}$  and  $p \in \mathbb{N}$ , we denote

$$\langle a \rangle_p = a(a-1)(a-2) \cdots (a-p+1). \quad (2.35)$$

Moreover, by (2.28), we define

$$\eta_i^a = (x_{i+1,i} + \sum_{j=1}^{i-1} x_{i+1,j} \partial_{i,j})^a = \sum_{p=0}^{\infty} \frac{\langle a \rangle_p}{p!} x_{i+1,i}^{a-p} \left( \sum_{j=1}^{i-1} x_{i+1,j} \partial_{i,j} \right)^p \quad (2.36)$$

as differential operators on  $\mathcal{A}_1$ , for  $i = 1, 2, \dots, n-1$  and  $a \in \mathbb{C}$ . If  $a \notin \mathbb{N}$ , then the above summation is infinite and the positions of  $x_{i+1,i}$  and  $(\sum_{j=1}^{i-1} x_{i+1,j} \partial_{i,j})$  are not symmetric. Since  $x_{i+1,i}$  and  $(\sum_{j=1}^{i-1} x_{i+1,j} \partial_{i,j})$  commute, we have

$$\eta_i^{a_1} \eta_i^{a_2} = \eta_i^{a_1+a_2} \quad \text{for } a_1, a_2 \in \mathbb{C}. \quad (2.37)$$

In particular, the inverse of the differential operator  $\eta_i^a$  is exactly  $\eta_i^{-a}$ .

Given two differential operators  $d$  and  $\bar{d}$ , we define the commutator

$$[d, \bar{d}] = d\bar{d} - \bar{d}d. \quad (2.38)$$

For any element  $f \in \mathcal{A}_1$  and  $r \in \mathbb{C}$ , we have

$$[\partial_{i+1,i}, x_{i+1,i}^r](f) = \partial_{i+1,i}(x_{i+1,i}^r f) - x_{i+1,i}^r \partial_{i+1,i}(f) = r x_{i+1,i}^{r-1} f, \quad (2.39)$$

that is,

$$[\partial_{i+1,i}, x_{i+1,i}^r] = r x_{i+1,i}^{r-1} \quad \text{as operators.} \quad (2.40)$$

Note that if  $(r, s) \notin \{(i+1, j) \mid j = 1, \dots, i\}$  and  $(p, q) \notin \{(i, j) \mid j = 1, \dots, i-1\}$ , then

$$[\partial_{r,s}, \eta_i^a] = [x_{p,q}, \eta_i^a] = 0 \quad (2.41)$$

directly by (2.36). Now

$$\begin{aligned} [\partial_{i+1,i}, \eta_i^a] &= \sum_{p=0}^{\infty} \frac{\langle a \rangle_p}{p!} [\partial_{i+1,i}, x_{i+1,i}^{a-p} (\sum_{j=1}^{i-1} x_{i+1,j} \partial_{i,j})^p] \\ &= \sum_{p=0}^{\infty} \frac{\langle a \rangle_p}{p!} (a-p) x_{i+1,i}^{a-p-1} (\sum_{j=1}^{i-1} x_{i+1,j} \partial_{i,j})^p \\ &= \sum_{p=0}^{\infty} \frac{a \langle a-1 \rangle_p}{p!} x_{i+1,i}^{a-p-1} (\sum_{j=1}^{i-1} x_{i+1,j} \partial_{i,j})^p = a \eta_i^{a-1} \end{aligned} \quad (2.42)$$

by (2.40). Moreover, for  $j = 1, 2, \dots, i-1$ ,

$$\begin{aligned} [\partial_{i+1,j}, \eta_i^a] &= \sum_{p=0}^{\infty} \frac{\langle a \rangle_p}{p!} [\partial_{i+1,j}, x_{i+1,i}^{a-p} (\sum_{s=1}^{i-1} x_{i+1,s} \partial_{i,s})^p] \\ &= \sum_{p=0}^{\infty} \frac{\langle a \rangle_p}{p!} p x_{i+1,i}^{a-p-1} (\sum_{s=1}^{i-1} x_{i+1,s} \partial_{i,s})^{p-1} \partial_{i,j} \\ &= \sum_{p=0}^{\infty} \frac{a \langle a-1 \rangle_{p-1}}{(p-1)!} x_{i+1,i}^{a-p-1} (\sum_{s=1}^{i-1} x_{i+1,s} \partial_{i,s})^{p-1} \partial_{i,j} = a \eta_i^{a-1} \partial_{i,j} \end{aligned} \quad (2.43)$$

and similarly,

$$[x_{i,j}, \eta_i^a] = -a \eta_i^{a-1} x_{i+1,j} \quad \text{for } j = 1, 2, \dots, i-1. \quad (2.44)$$

**Lemma 2.2.** For  $i, l \in \{1, 2, \dots, n-1\}$  and  $a \in \mathbb{C}$ , we have:

$$[d_l, \eta_i^a] = a \delta_{i,l} \eta_i^{a-1} (1 - a + \zeta_i). \quad (2.45)$$

*Proof.* Note that

$$[E_{l,l+1}, E_{i+1,i}^m] = m \delta_{i,l} E_{i+1,i}^{m-1} (1 - m + h_i) \quad \text{for } m \in \mathbb{N} \quad (2.46)$$

(cf. (2.3)). So (2.45) holds for any  $a \in \mathbb{N}$  by (2.25), (2.28) and (2.30). Since (2.45) is completely determined by (2.41)-(2.44), which are independent of whether  $a$  is a nonnegative integer, it must hold for any  $a \in \mathbb{C}$ .  $\square$

Denote the Cartan matrix of  $sl(n, \mathbb{C})$  by

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n-1} \\ \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{bmatrix} = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}. \quad (2.47)$$



**Lemma 2.3.** For  $i, l \in \{1, 2, \dots, n-1\}$  and  $a \in \mathbb{C}$ ,

$$[\zeta_l, \eta_i^a] = -aa_{l,i}\eta_i^a. \quad (2.48)$$

*Proof.* Observe that

$$[h_l, E_{i+1,i}^m] = ma_{l,i}E_{i+1,i}^m \quad \text{for } m \in \mathbb{N} \quad (2.49)$$

(cf. (2.3)). Hence (2.48) holds for any  $a \in \mathbb{N}$  by (2.28) and (2.30). Again (2.48) is completely determined by (2.41)-(2.44), which are independent of whether  $a$  is a nonnegative integer. Thus (2.48) must hold for any  $a \in \mathbb{C}$ .  $\square$

In order to construct a differential-operator representation of the symmetric group  $S_n$  on  $\mathcal{A}_1$ , we need the following result.

**Lemma 2.4.** For any  $a_1, a_2 \in \mathbb{C}$  and  $1 \leq i < n-1$ , we have

$$\eta_i^{a_1}\eta_{i+1}^{a_1+a_2}\eta_i^{a_2} = \eta_{i+1}^{a_2}\eta_i^{a_1+a_2}\eta_{i+1}^{a_1}. \quad (2.50)$$

*Proof.* Note that for  $a \in \mathbb{C}$ , we have

$$\left[\sum_{p=1}^i x_{i+2,p}\partial_{i+1,p}, x_{i+1,i}^a\right] = ax_{i+1,i}^{a-1}x_{i+2,i} \quad (2.51)$$

by (2.40). Moreover,

$$\left[\sum_{p=1}^i x_{i+2,p}\partial_{i+1,p}, \sum_{j=1}^{i-1} x_{i+1,j}\partial_{i,j}\right] = \sum_{j=1}^{i-1} x_{i+2,j}\partial_{i,j}. \quad (2.52)$$

Hence

$$\begin{aligned} & \eta_i^{a_1}\eta_{i+1}^{a_1+a_2} \\ &= \sum_{p,q=0}^{\infty} \frac{\langle a_1 \rangle_p \langle a_1 + a_2 \rangle_q}{p!q!} x_{i+2,i+1}^{a_1+a_2-q} x_{i+1,i}^{a_1-p} \left(\sum_{j_1=1}^{i-1} x_{i+1,j_1}\partial_{i,j_1}\right)^p \left(\sum_{j_2=1}^i x_{i+2,j_2}\partial_{i+1,j_2}\right)^q \\ &= \sum_{p,q,r,s=0}^{\infty} \frac{(-1)^{r+s} \langle a_1 \rangle_{p+r} \langle a_1 + a_2 \rangle_q \langle p \rangle_s \langle q \rangle_{r+s}}{r!s!p!q!} x_{i+2,i+1}^{a_1+a_2-q} \left(\sum_{j_2=1}^i x_{i+2,j_2}\partial_{i+1,j_2}\right)^{q-r-s} x_{i+1,i}^{a_1-p-r} \\ & \quad \times \left(\sum_{j_1=1}^{i-1} x_{i+1,j_1}\partial_{i,j_1}\right)^{p-s} x_{i+2,i}^r \left(\sum_{j=1}^{i-1} x_{i+2,j}\partial_{i,j}\right)^s \\ &= \sum_{p,q,r,s=0}^{\infty} \frac{(-1)^{r+s} \langle a_1 \rangle_{p+r} \langle a_1 + a_2 \rangle_q}{r!s!(p-s)!(q-r-s)!} x_{i+2,i+1}^{a_1+a_2-q} \left(\sum_{j_2=1}^i x_{i+2,j_2}\partial_{i+1,j_2}\right)^{q-r-s} x_{i+1,i}^{a_1-p-r} \\ & \quad \times \left(\sum_{j_1=1}^{i-1} x_{i+1,j_1}\partial_{i,j_1}\right)^{p-s} x_{i+2,i}^r \left(\sum_{j=1}^{i-1} x_{i+2,j}\partial_{i,j}\right)^s \end{aligned}$$

$$\begin{aligned}
&= \sum_{q,k,s=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^k \langle a_1 \rangle_k \langle a_1 - k \rangle_{p-s} \langle a_1 + a_2 \rangle_q}{(k-s)! s! (p-s)! (q-k)!} x_{i+2,i+1}^{a_1+a_2-q} \left( \sum_{j_2=1}^i x_{i+2,j_2} \partial_{i+1,j_2} \right)^{q-k} \\
&\quad \times x_{i+1,i}^{a_1-k-(p-s)} \left( \sum_{j_1=1}^{i-1} x_{i+1,j_1} \partial_{i,j_1} \right)^{p-s} x_{i+2,i}^{k-s} \left( \sum_{j=1}^{i-1} x_{i+2,j} \partial_{i,j} \right)^s \\
&= \sum_{q,k,s=0}^{\infty} \frac{(-1)^k \langle a_1 \rangle_k \langle a_1 + a_2 \rangle_q}{(k-s)! s! (q-k)!} x_{i+2,i+1}^{a_1+a_2-q} \left( \sum_{j_2=1}^i x_{i+2,j_2} \partial_{i+1,j_2} \right)^{q-k} \\
&\quad \times \eta_i^{a_1-k} x_{i+2,i}^{k-s} \left( \sum_{j=1}^{i-1} x_{i+2,j} \partial_{i,j} \right)^s \\
&= \sum_{q,k=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^k \langle a_1 \rangle_k \langle a_1 + a_2 \rangle_q}{(k-s)! s! (q-k)!} x_{i+2,i+1}^{a_1+a_2-q} \left( \sum_{j_2=1}^i x_{i+2,j_2} \partial_{i+1,j_2} \right)^{q-k} \\
&\quad \times \eta_i^{a_1-k} x_{i+2,i}^{k-s} \left( \sum_{j=1}^{i-1} x_{i+2,j} \partial_{i,j} \right)^s \\
&= \sum_{q,k=0}^{\infty} \frac{(-1)^k \langle a_1 \rangle_k \langle a_1 + a_2 \rangle_q}{k! (q-k)!} x_{i+2,i+1}^{a_1+a_2-q} \left( \sum_{j_2=1}^i x_{i+2,j_2} \partial_{i+1,j_2} \right)^{q-k} \\
&\quad \times \eta_i^{a_1-k} \left( x_{i+2,i} + \sum_{j=1}^{i-1} x_{i+2,j} \partial_{i,j} \right)^k \\
&= \sum_{k=0}^{\infty} \sum_{q=0}^{\infty} \frac{(-1)^k \langle a_1 \rangle_k \langle a_1 + a_2 \rangle_k \langle a_1 + a_2 - k \rangle_{q-k}}{k! (q-k)!} x_{i+2,i+1}^{a_1+a_2-k-(q-k)} \left( \sum_{j_2=1}^i x_{i+2,j_2} \partial_{i+1,j_2} \right)^{q-k} \\
&\quad \times \eta_i^{a_1-k} \left( x_{i+2,i} + \sum_{j=1}^{i-1} x_{i+2,j} \partial_{i,j} \right)^k \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k \langle a_1 \rangle_k \langle a_1 + a_2 \rangle_k}{k!} \eta_{i+1}^{a_1+a_2-k} \eta_i^{a_1-k} \left( x_{i+2,i} + \sum_{j=1}^{i-1} x_{i+2,j} \partial_{i,j} \right)^k \tag{2.53}
\end{aligned}$$

by (2.36), (2.51) and (2.52). Similarly, we have

$$\eta_i^{a_1+a_2} \eta_{i+1}^{a_1} = \sum_{k=0}^{\infty} \frac{(-1)^k \langle a_1 \rangle_k \langle a_1 + a_2 \rangle_k}{k!} \eta_{i+1}^{a_1-k} \eta_i^{a_1+a_2-k} \left( x_{i+2,i} + \sum_{j=1}^{i-1} x_{i+2,j} \partial_{i,j} \right)^k. \tag{2.54}$$

Thus

$$\begin{aligned}
\eta_i^{a_1} \eta_{i+1}^{a_1+a_2} \eta_i^{a_2} &= \sum_{k=0}^{\infty} \frac{(-1)^k \langle a_1 \rangle_k \langle a_1 + a_2 \rangle_k}{k!} \eta_{i+1}^{a_1+a_2-k} \eta_i^{a_1+a_2-k} \left( x_{i+2,i} + \sum_{j=1}^{i-1} x_{i+2,j} \partial_{i,j} \right)^k \\
&= \eta_{i+1}^{a_2} \eta_i^{a_1+a_2} \eta_{i+1}^{a_1}. \quad \square \tag{2.55}
\end{aligned}$$

It is well known that the symmetric group  $S_n$  is a group generated by  $\{\sigma_1, \dots, \sigma_{n-1}\}$  with the defining relations:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_r \sigma_s = \sigma_s \sigma_r, \quad \sigma_r^2 = 1 \tag{2.56}$$

for  $i = 1, 2, \dots, n-2$  and  $r, s = 1, 2, \dots, n-1$  such that  $|r-s| \geq 2$ . According to (2.31) and (2.34), any element  $f \in \mathcal{A}_1$  can be written as  $f = \sum_{j \in \mathbb{Z}} f_j$  such that

$$\zeta_i(f_j) = \mu_{(j)}(h_i)f_j, \quad \mu_{(j)} \in H^*. \quad (2.57)$$

We define an action of  $\{\sigma_1, \dots, \sigma_{n-1}\}$  on  $\mathcal{A}_1$  by

$$\sigma_i(f) = \sum_{j \in \mathbb{Z}} \eta_i^{\mu_{(j)}(h_i)+1}(f_j), \quad i = 1, 2, \dots, n-1. \quad (2.58)$$

**Theorem 2.5.** *Expression (2.58) gives a representation of the symmetric group  $S_n$ . Moreover,  $\{\sigma(1) \mid \sigma \in S_n\}$  are weighted solutions of the system (2.27) of partial differential equations.*

*Proof.* Let  $f \in \mathcal{A}_1$  be a weight function with weight  $\mu$ , that is,  $\zeta_i(f) = \mu(h_i)f$  for  $i = 1, 2, \dots, n-1$ . Then

$$(\sigma_i|_{\mathcal{A}_1})^2(f) = \sigma_i(\eta_i^{\mu(h_i)+1}(f)) = \eta_i^{-\mu(h_i)-1}(\eta_i^{\mu(h_i)+1}(f)) = f \quad (2.59)$$

by (2.37), (2.47), (2.48) and (2.58). Thus

$$(\sigma_i|_{\mathcal{A}_1})^2 = \text{Id}_{\mathcal{A}_1} \quad \text{for } i = 1, 2, \dots, n-1. \quad (2.60)$$

Note

$$\begin{aligned} & [(\sigma_i|_{\mathcal{A}_1})(\sigma_{i+1}|_{\mathcal{A}_1})(\sigma_i|_{\mathcal{A}_1})](f) \\ &= \sigma_i[\sigma_{i+1}(\eta_i^{\mu(h_i)+1}(f))] = \sigma_i[\eta_{i+1}^{\mu(h_{i+1})+\mu(h_i)+2}(\eta_i^{\mu(h_i)+1}(f))] \\ &= \eta_i^{\mu(h_{i+1})+1}[\eta_{i+1}^{\mu(h_{i+1})+\mu(h_i)+2}(\eta_i^{\mu(h_i)+1}(f))] = (\eta_i^{\mu(h_{i+1})+1}\eta_{i+1}^{\mu(h_{i+1})+\mu(h_i)+2}\eta_i^{\mu(h_i)+1})(f) \\ &= (\eta_{i+1}^{\mu(h_i)+1}\eta_i^{\mu(h_{i+1})+\mu(h_i)+2}\eta_{i+1}^{\mu(h_{i+1})+1})(f) = \eta_{i+1}^{\mu(h_i)+1}[\eta_i^{\mu(h_{i+1})+\mu(h_i)+2}(\eta_{i+1}^{\mu(h_{i+1})+1}(f))] \\ &= [(\sigma_{i+1}|_{\mathcal{A}_1})(\sigma_i|_{\mathcal{A}_1})(\sigma_{i+1}|_{\mathcal{A}_1})](f) \end{aligned} \quad (2.61)$$

by (2.47), (2.48), (2.50) and (2.58). Hence

$$(\sigma_i|_{\mathcal{A}_1})(\sigma_{i+1}|_{\mathcal{A}_1})(\sigma_i|_{\mathcal{A}_1}) = (\sigma_{i+1}|_{\mathcal{A}_1})(\sigma_i|_{\mathcal{A}_1})(\sigma_{i+1}|_{\mathcal{A}_1}). \quad (2.62)$$

For  $|r-s| \geq 2$ , we have

$$\eta_r^a \eta_s^b = (x_{r+1,r} + \sum_{j=1}^{r-1} x_{r+1,j} \partial_{r,j})^a (x_{s+1,s} + \sum_{j=1}^{s-1} x_{s+1,j} \partial_{s,j})^b = \eta_s^b \eta_r^a \quad (2.63)$$

for  $a, b \in \mathbb{C}$  by (2.36). So

$$[(\sigma_r|_{\mathcal{A}_1})(\sigma_s|_{\mathcal{A}_1})](f) = (\eta_r^{\mu(h_r)+1} \eta_s^{\mu(h_s)+1})(f) = [(\sigma_s|_{\mathcal{A}_1})(\sigma_r|_{\mathcal{A}_1})](f), \quad (2.64)$$

which implies

$$(\sigma_r|_{\mathcal{A}_1})(\sigma_s|_{\mathcal{A}_1}) = (\sigma_s|_{\mathcal{A}_1})(\sigma_r|_{\mathcal{A}_1}). \quad (2.65)$$

According to (2.56), this proves that (2.58) defines a representation of  $S_n$  on  $\mathcal{A}_1$ .

Now we assume that  $f \in \mathcal{A}_1$  is a weighted solution of (2.27) with weight  $\mu$ , that is,  $d_i(f) = 0$  for  $i = 1, 2, \dots, n-1$ . Given  $r \in \{1, 2, \dots, n-1\}$ ,

$$\begin{aligned} d_i(\sigma_r(f)) &= d_i \eta_r^{\mu(h_r)+1}(f) = [d_i, \eta_r^{\mu(h_r)+1}](f) \\ &= (\mu(h_r) + 1) \delta_{i,r} \eta_r^{\mu(h_r)} (-\mu(h_r) + \zeta_r)(f) = 0 \end{aligned} \quad (2.66)$$

by (2.45). So  $\sigma_r(f)$  is also a weighted solution of (2.27). Recall that  $S_n$  is generated by  $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ . For any  $\sigma \in S_n$ , we write  $\sigma = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_r}$  and

$$\sigma(1) = \sigma_{i_1}(\sigma_{i_2}(\cdots(\sigma_{i_r}(1))\cdots)) \quad (2.67)$$

is a weighted solution of (2.27) by (2.66) and induction on  $r$ .  $\square$

### 3 Completeness

In this section, we want to prove the following theorem of completeness:

**Theorem 3.1.** *The solution space of the system (2.27) in  $\mathcal{A}_1$  is spanned by  $\{\sigma(1) \mid \sigma \in S_n\}$ . Moreover,  $\{\sigma(1) \mid \sigma \in S_n\}$  are all the weighted solutions of the system (2.27) in  $\mathcal{A}_1$  up to scalar multiples. In particular, there are exactly  $n!$  singular vectors up to scalar multiples in the Verma module  $M_\lambda$  when the weight  $\lambda$  is dominant integral.*

*Proof.* For any

$$\vec{a} = (a_1, a_2, \dots, a_{n-1}) \in \mathbb{C}^{n-1}, \quad (3.1)$$

we define

$$\begin{aligned} \phi_{\vec{a}} &= \eta_2^{a_2} \eta_1^{a_2 - \lambda_{n-1} - 1} \cdots \eta_i^{a_i} \eta_{i-1}^{a_i - \lambda_{n-1} - 1} \cdots \eta_1^{a_i - \lambda_{n-1} - \cdots - \lambda_{n-(i-1)} - (i-1)} \\ &\quad \cdots \eta_{n-1}^{a_{n-1}} \eta_{n-2}^{a_{n-1} - \lambda_{n-1} - 1} \cdots \eta_1^{a_{n-1} + 2 - \lambda_2 - \cdots - \lambda_{n-1} - n}(1). \end{aligned} \quad (3.2)$$

Then

$$\phi_{\vec{a}} \in \mathcal{A}_1 \quad (3.3)$$

(cf. (2.32)-(2.34)) and is a solution of the system

$$d_2(z) = d_3(z) = \cdots = d_{n-1}(z) = 0 \quad (3.4)$$

by Lemmas 2.2, 2.3 and (2.66).

**Claim.** An element  $z$  in  $\mathcal{A}_1$  is a solution of the system (3.4) if and only if it can be written as

$$z = \sum_{\vec{j} \in \mathbb{N}^{n-1}} \sum_{i=1}^p c_{\vec{a}^i - \vec{j}} x_{2,1}^{a_1^i - j_1} \phi_{\vec{a}^i - \vec{j}} \quad \text{with } c_{\vec{a}^i - \vec{j}} \in \mathbb{C} \quad (3.5)$$

for some  $\vec{a}^1, \dots, \vec{a}^p \in \mathbb{C}^{n-1}$ .

Recall  $\eta_1 = x_{2,1}$ . By Lemma 2.2, the sufficiency holds. Now we want to prove the necessity. Recall that

$$E_{i+1,i} = d_i, \quad E_{i,i+1} = \eta_i \quad \text{as operators on } \mathcal{A}_1 \quad (3.6)$$

(cf. (2.25) and (2.28)) for  $i = 1, 2, \dots, n-1$ . Note

$$d_{n-1} = (\lambda_{n-1} - x_{n,n-1} \partial_{n,n-1}) \partial_{n,n-1} + \sum_{i=1}^{n-2} x_{n-1,i} \partial_{n,i}. \quad (3.7)$$

Moreover, (2.11) tells us that

$$\begin{aligned} d_{n-2,n} &= E_{n-2,n}|_{\mathcal{A}_1} \\ &= (\lambda_{n-1} + \lambda_{n-2} - x_{n,n-2} \partial_{n,n-2} - x_{n,n-1} \partial_{n,n-1}) \partial_{n,n-2} \\ &\quad - d_{n-1} \partial_{n-1,n-2} + \sum_{i=1}^{n-3} x_{n-2,i} \partial_{n,i} \\ &= (\lambda_{n-1} + \lambda_{n-2} + 1 - x_{n,n-2} \partial_{n,n-2} - x_{n,n-1} \partial_{n,n-1}) \partial_{n,n-2} \\ &\quad - \partial_{n-1,n-2} d_{n-1} + \sum_{i=1}^{n-3} x_{n-2,i} \partial_{n,i}. \end{aligned} \quad (3.8)$$

Set

$$\bar{\lambda}_i = n - i - 1 + \sum_{p=i}^{n-1} \lambda_p \quad \text{for } i = 2, 3, \dots, n-2. \quad (3.9)$$

Furthermore,

$$\begin{aligned} d_{n-3,n} &= E_{n-3,n}|_{\mathcal{A}_1} \\ &= (\bar{\lambda}_{n-3} - 2 - x_{n,n-3} \partial_{n,n-3} - x_{n,n-2} \partial_{n,n-2} - x_{n,n-1} \partial_{n,n-1}) \partial_{n,n-3} \\ &\quad - d_{n-2,n} \partial_{n-2,n-3} - d_{n-1} \partial_{n-1,n-3} + \sum_{i=1}^{n-4} x_{n-3,i} \partial_{n,i} \\ &= (\bar{\lambda}_{n-2} - 2 - \sum_{p=1}^3 x_{n,n-p} \partial_{n,n-p}) \partial_{n,n-3} - \partial_{n-2,n-3} d_{n-2,n} - \partial_{n-1,n-3} d_{n-1} \\ &\quad - [d_{n-2,n}, \partial_{n-2,n-3}] - [d_{n-1}, \partial_{n-1,n-3}] + \sum_{i=1}^{n-4} x_{n-3,i} \partial_{n,i} \\ &= (\bar{\lambda}_{n-3} - \sum_{p=1}^3 x_{n,n-p} \partial_{n,n-p}) \partial_{n,n-3} + \sum_{i=1}^{n-4} x_{n-3,i} \partial_{n,i} \\ &\quad - \partial_{n-2,n-3} d_{n-2,n} - \partial_{n-1,n-3} d_{n-1}. \end{aligned} \quad (3.10)$$

By induction, we can prove that

$$d_{i,n} = E_{i,n}|_{\mathcal{A}_1} = (\bar{\lambda}_i - \sum_{p=i}^{n-1} x_{n,p} \partial_{n,p}) \partial_{n,i} + \sum_{q=1}^{i-1} x_{i,q} \partial_{n,q} - \sum_{j=i+1}^{n-1} \partial_{j,i} d_{j,n} \quad (3.11)$$

for  $i = 2, 3, \dots, n-2$ , where we take

$$d_{n-1,n} = d_{n-1}. \quad (3.12)$$

Suppose that

$$z = f x_{2,1}^{a_1} \phi_{\vec{a}} \quad (3.13)$$

is a solution of the system (3.4) for some  $\vec{a} \in \mathbb{C}^{n-1}$  and  $f \in \mathcal{A}_0$  (cf. (2.32) and (3.2)). We want to prove that  $f$  is a constant. Denote

$$\epsilon_i = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0) \in \mathbb{C}^{n-1} \quad (3.14)$$

and

$$\iota_{i,j} = a_{i+j} - \sum_{p=1}^j (\lambda_{n-p} + 1) \quad \text{for } 2 \leq i \leq n-1, 0 \leq j \leq n-i-1. \quad (3.15)$$

We define

$$U_i = \left\{ \sum_{\vec{j} \in \mathbb{N}^{n-1}} g_{\vec{j}} x_{2,1}^{a_1 - j_1} \phi_{\vec{a} - \epsilon_i - \vec{j}} \mid g_{\vec{j}} \in \mathcal{A}_0 \right\} \quad (3.16)$$

for  $i = 2, \dots, n-1$ , and

$$U = \sum_{i=2}^{n-1} U_i. \quad (3.17)$$

Moreover, for fixed  $i \geq 2$ ,

$$\{\eta_i^{\iota_{i,j}} \mid 0 \leq j \leq n-i-1\} \quad (3.18)$$

are all the factors in the righthand side of (3.2) that contain  $x_{i+1,p}$  or  $\partial_{i,q}$  with  $p = 1, \dots, i$  and  $q = 1, 2, \dots, i-1$  by (2.36). Besides,

$$[\partial_{i+1,i}, \eta_i^{\iota_{i,j}}] = \iota_{i,j} \eta_i^{\iota_{i,j}-1}, \quad (3.19)$$

$$[\partial_{i+1,p}, \eta_i^{\iota_{i,j}}] = \iota_{i,j} \eta_i^{\iota_{i,j}-1} \partial_{i,p} \quad \text{for } p = 1, \dots, i-1, \quad (3.20)$$

$$\left[ \sum_{p=r}^i x_{i+1,p} \partial_{i+1,p}, \eta_i^{\iota_{i,j}} \right] = \iota_{i,j} (\eta_i^{\iota_{i,j}} - \sum_{p=1}^{r-1} x_{i+1,p} \eta_i^{\iota_{i,j}-1} \partial_{i,p}), \quad (3.21)$$

$$\left[ \sum_{p=q}^{i-1} x_{i,p} \partial_{i,p}, \eta_i^{\iota_{i,j}} \right] = -\iota_{i,j} \sum_{p=q}^{i-1} x_{i+1,p} \eta_i^{\iota_{i,j}-1} \partial_{i,p}, \quad (3.22)$$

where  $2 \leq r \leq i$  and  $2 \leq q \leq i-1$ .

By (3.18)-(3.22), we have

$$\partial_{i+1,r}(\phi_{\bar{a}}) \in U_i \quad (3.23)$$

and

$$\left(\sum_{p=r}^i x_{i+1,p} \partial_{i+1,p}\right)(\phi_{\bar{a}}) \equiv c_{i,r} \phi_{\bar{a}} \pmod{\sum_{s=1}^i U_s}, \quad c_{i,r} \in \mathbb{C}, \quad (3.24)$$

for  $2 \leq i \leq n-1$  and  $1 \leq r \leq i$ . Since for  $2 \leq i \leq n-2$ ,

$$E_{i,n} = [E_{i,i+1}, [E_{i+1,i+2}, \dots, [E_{n-2,n-1}, E_{n-1,n}] \dots]], \quad (3.25)$$

we have

$$d_{i,n} = [d_i, [d_{i+1}, \dots [d_{n-2}, d_{n-1}] \dots]]. \quad (3.26)$$

Thus

$$d_{i,n}(z) = 0 \quad \text{for } 2 \leq i \leq n-1. \quad (3.27)$$

By (3.11), (3.16) and (3.22)-(3.27), we obtain

$$d_{i,n}(z) = [(\bar{\lambda}_i - c_{n-1,i}) \partial_{n,i}(f) + \sum_{q=1}^{i-1} x_{i,q} \partial_{n,q}(f)] x_{2,1}^{a_1} \phi_{\bar{a}} - \sum_{j=i+1}^{n-1} \partial_{j,i} d_{j,n}(z) \equiv 0 \pmod{U} \quad (3.28)$$

for  $i = 2, 3, \dots, n-2$  and

$$d_{n-1}(z) = \left(\sum_{q=1}^{n-2} x_{n-1,q} \partial_{n,q}(f)\right) x_{2,1}^{a_1} \phi_{\bar{a}} \equiv 0 \pmod{U}. \quad (3.29)$$

Since the constraint on  $d_{r,n}(z) \equiv 0 \pmod{U}$  for  $r \geq 2$  implies  $\partial_{r,s} d_{r,n}(z) \equiv 0 \pmod{U}$  for  $s = 1, 2, \dots, r-1$  by (3.23), (2.28) is equivalent to

$$d_{i,n}(z) = [(\bar{\lambda}_i - c_{n-1,i}) \partial_{n,i}(f) + \sum_{q=1}^{i-1} x_{i,q} \partial_{n,q}(f)] x_{2,1}^{a_1} \phi_{\bar{a}} \equiv 0 \pmod{U} \quad (3.30)$$

for  $i = 2, 3, \dots, n-2$ .

Expressions (3.29) and (3.30) give

$$\sum_{q=1}^{i-1} x_{i,q} \partial_{n,q}(f) + (\bar{\lambda}_i - c_{n-1,i}) \partial_{n,i}(f) = 0 \quad \text{for } i = 2, \dots, n-2, \quad (3.31)$$

$$\sum_{q=1}^{n-2} x_{n-1,q} \partial_{n,q}(f) = 0. \quad (3.32)$$

We view

$$\partial_{n,1}(f), \partial_{n,2}(f), \dots, \partial_{n,n-2}(f) \quad \text{as unknowns.} \quad (3.33)$$

Then the coefficient determinant of the system (3.31) and (3.32) is

$$\begin{aligned}
& \begin{vmatrix} x_{2,1} & \lambda_2 - c_{n-1,2} & & \\ x_{3,1} & x_{3,2} & \ddots & \\ \vdots & \ddots & \ddots & \lambda_{n-2} - c_{n-1,n-2} \\ x_{n-1,a} & \cdots & x_{n-1,n-3} & x_{n-1,n-2} \end{vmatrix} \\
&= \prod_{p=2}^{n-1} x_{p,p-1} + g(x_{2,1}, x_{3,2}, \dots, x_{n-1,n-2}) \neq 0,
\end{aligned} \tag{3.34}$$

where  $g(x_{2,1}, x_{3,2}, \dots, x_{n-1,n-2})$  is a polynomial of degree  $n-3$  in  $\{x_{2,1}, x_{3,2}, \dots, x_{n-1,n-2}\}$  over  $\mathcal{A}_0$  (cf. (2.32)). Therefore,

$$\partial_{n,q}(f) = 0 \quad \text{for } q = 1, 2, \dots, n-2. \tag{3.35}$$

Based on our calculations in (3.23)-(3.25), we can prove by induction that

$$\partial_{q+r,q}(f) = 0 \quad \text{for } 1 \leq q \leq n-2, 2 \leq r \leq n-q. \tag{3.36}$$

So  $f$  is a constant.

Suppose that  $z$  is any solution of the system (3.4) in  $\mathcal{A}_1$ . By (2.34) and (3.2), it can be written as

$$z = \sum_{\vec{j} \in \mathbb{N}^{n-1}} \sum_{i=1}^p f_{\vec{a}^i - \vec{j}} x_{2,1}^{a_1^i - j_1} \phi_{\vec{a}^i - \vec{j}} \quad \text{with } f_{\vec{j}} \in \mathcal{A}_0. \tag{3.37}$$

Let

$$\mathcal{S} = \{\vec{b} \in \mathbb{C}^{n-1} \mid f_{\vec{b}} \neq 0; f_{\vec{b} + \vec{j}} = 0 \text{ for all } \vec{0} \neq \vec{j} \in \mathbb{N}^{n-1}\}. \tag{3.38}$$

The above arguments show that

$$\{f_{\vec{b}} \mid \vec{b} \in \mathcal{S}\} \quad \text{are constants} \tag{3.39}$$

(cf. the key equations (3.29) and (3.30)). Since  $\sum_{\vec{b} \in \mathcal{S}} f_{\vec{b}} x_{2,1}^{b_1} \phi_{\vec{b}}$  is a solution of the system (3.4), so is  $z - \sum_{\vec{b} \in \mathcal{S}} f_{\vec{b}} x_{2,1}^{b_1} \phi_{\vec{b}}$ . By induction, we prove the Claim

To solve the system (2.27) in  $\mathcal{A}_1$ , we only need to consider the solutions of the form  $z = x_{2,1}^{a_1} \phi_{\vec{a}}$  with  $\vec{a} \in \mathbb{C}^{n-1}$  by the above claim, because (2.27) is a weighted system. Note

$$d_1 = (\lambda_1 - \sum_{j=2}^n x_{j,1} \partial_{j,1} + \sum_{j=3}^n x_{j,2} \partial_{j,2}) \partial_{2,1} - \sum_{j=3}^n x_{j,2} \partial_{j,1}. \tag{3.40}$$

Denote

$$\hat{a}_{1,r} = \sum_{p=r}^{n-1} a_i - \sum_{p=2}^{n-r} (p-1)(\lambda_p + 1) - (n-r) \sum_{q=n-r+1}^{n-1} (\lambda_q + 1), \quad \hat{a}_{1,1} = a_1 + \hat{a}_{1,2} \tag{3.41}$$



for  $r = 2, 3, \dots, n-1$ ,

$$\hat{a}_{2,r} = a_r - \sum_{p=1}^{r-2} (\lambda_{n-p} + 1) \quad \text{for } r = 2, 3, \dots, n-1. \quad (3.42)$$

and

$$\tilde{a} = \sum_{i=2}^{n-1} a_i - \sum_{p=3}^{n-1} (p-2)(\lambda_p + 1). \quad (3.43)$$

Letting  $x_{p,q} = 0$  for  $1 \leq q \leq p-2 \leq n-2$  in

$$d_1(z) = [(\lambda_1 - \sum_{j=2}^n x_{j,1} \partial_{j,1} + \sum_{j=3}^n x_{j,2} \partial_{j,2}) \partial_{2,1} - \sum_{j=3}^n x_{j,2} \partial_{j,1}] (x_{2,1}^{a_1} \phi_{\vec{a}}) = 0, \quad (3.44)$$

we get

$$\hat{a}_{1,1}(\lambda_1 + 1 - \hat{a}_{1,1} + \tilde{a}) - \sum_{r=2}^{n-1} \hat{a}_{2,r} \hat{a}_{1,r} = 0 \quad (3.45)$$

by (2.36) and (3.2).

Suppose  $n > 3$ . We take  $x_{p,q} = 0$  for  $1 \leq q \leq p-2 \leq n-2$  in

$$\partial_{n,1} d_1(z) = \partial_{n,1} [(\lambda_1 - \sum_{j=2}^n x_{j,1} \partial_{j,1} + \sum_{j=3}^n x_{j,2} \partial_{j,2}) \partial_{2,1} - \sum_{j=3}^n x_{j,2} \partial_{j,1}] (x_{2,1}^{a_1} \phi_{\vec{a}}) = 0, \quad (3.46)$$

and obtain

$$\begin{aligned} & a_{n-1} \left[ \prod_{i=1}^{n-2} (a_{n-1} - i - \sum_{p=1}^i \lambda_{n-p}) \right] [(\hat{a}_{1,1} - 1)(\lambda_1 - \hat{a}_{1,1} + \tilde{a}) \\ & - \sum_{r=2}^{n-2} \hat{a}_{2,r} (\hat{a}_{1,r} - 1) - (\hat{a}_{2,n-1} - 1)(\hat{a}_{1,n-1} - 1)] = 0. \end{aligned} \quad (3.47)$$

Note that

$$\begin{aligned} & [\hat{a}_{1,1}(\lambda_1 + 1 - \hat{a}_{1,1} + \tilde{a}) - \sum_{r=2}^{n-1} \hat{a}_{2,r} \hat{a}_{1,r}] - [(\hat{a}_{1,1} - 1)(\lambda_1 - \hat{a}_{1,1} + \tilde{a}) \\ & - \sum_{r=2}^{n-2} \hat{a}_{2,r} (\hat{a}_{1,r} - 1) - (\hat{a}_{2,n-1} - 1)(\hat{a}_{1,n-1} - 1)] \\ & = \lambda_1 + \tilde{a} - \sum_{r=2}^{n-1} \hat{a}_{2,r} + 1 - \hat{a}_{1,n-1} = \lambda_1 + 1 - \hat{a}_{1,n-1} \\ & = (n-1) + \sum_{i=1}^{n-1} \lambda_i - a_{n-1}. \end{aligned} \quad (3.48)$$

By (3.45), (3.47) and (3.48), we have

$$a_{n-1} \prod_{i=1}^{n-1} (a_{n-1} - i - \sum_{p=1}^i \lambda_{n-p}) = 0. \quad (3.49)$$

Therefore,

$$a_{n-1} \in \{0, i + \sum_{p=1}^i \lambda_{n-p} \mid i = 1, 2, \dots, n-1\}. \quad (3.50)$$

Assume  $n = 3$ . Then

$$d_1 = (\lambda_1 - x_{2,1}\partial_{2,1} - x_{3,1}\partial_{3,1} + x_{3,2}\partial_{3,2})\partial_{2,1} - x_{3,2}\partial_{3,1}, \quad (3.51)$$

$$z = x_{2,1}^{a_1}\phi_{\vec{a}} = x_{2,1}^{a_1}(x_{3,2} + x_{3,1}\partial_{2,1})^{a_2}(x_{2,1}^{a_2-\lambda_2-1}), \quad (3.52)$$

and (3.45) becomes

$$(a_1 + a_2 - \lambda_2 - 1)(\lambda_1 + \lambda_2 + 2 - a_1) - a_2(a_2 - \lambda_2 - 1) = 0 \quad (3.53)$$

Letting  $x_{3,1} = 0$  in

$$\begin{aligned} \partial_{3,1}d_1(z) &= \partial_{3,1}[(\lambda_1 - x_{2,1}\partial_{2,1} - x_{3,1}\partial_{3,1} + x_{3,2}\partial_{3,2})\partial_{2,1} \\ &\quad - x_{3,2}\partial_{3,1}][x_{2,1}^{a_1}(x_{3,2} + x_{3,1}\partial_{2,1})^{a_2}(x_{2,1}^{a_2-\lambda_2-1})] = 0, \end{aligned} \quad (3.54)$$

we get

$$\begin{aligned} &a_2(a_2 - \lambda_2 - 1)(a_1 + a_2 - \lambda_2 - 2)(\lambda_1 + \lambda_2 + 1 - a_1) \\ &- a_2(a_2 - 1)(a_2 - \lambda_2 - 1)(a_2 - \lambda_2 - 2) = 0, \end{aligned} \quad (3.55)$$

equivalently

$$a_2(a_2 - \lambda_2 - 1)[(a_1 + a_2 - \lambda_2 - 2)(\lambda_1 + \lambda_2 + 1 - a_1) - (a_2 - 1)(a_2 - \lambda_2 - 2)] = 0. \quad (3.56)$$

By (3.53), we have

$$\begin{aligned} &(a_1 + a_2 - \lambda_2 - 2)(\lambda_1 + \lambda_2 + 1 - a_1) - (a_2 - 1)(a_2 - \lambda_2 - 2) \\ &= -(\lambda_1 + a_2) + a_2(a_2 - \lambda_2 - 1) - (a_2 - 1)(a_2 - \lambda_2 - 2) \\ &= -(\lambda_1 + a_2) + 2a_2 - \lambda_2 - 2 \\ &= a_2 - \lambda_1 - \lambda_2 - 2. \end{aligned} \quad (3.57)$$

Thus (3.56) and (3.57) give

$$a_2(a_2 - \lambda_2 - 1)(a_2 - \lambda_1 - \lambda_2 - 2) = 0, \quad (3.58)$$

which implies that (3.50) holds for any  $n \geq 2$ .

When  $n = 2$ , the solution space of (2.27) is  $\mathbb{C} + \mathbb{C}\sigma_1(1)$  by (2.66). In general, we can use (3.50) to reduce the problem of solving (2.27) to  $sl(n-1)$  as follows. Denote

$$\begin{aligned} \Psi_i &= x_{2,1}^{a_1}\eta_2^{a_2}\eta_1^{a_2-\lambda_{n-1}-1} \cdots \eta_r^{a_r}\eta_{r-1}^{a_r-\lambda_{n-1}-1} \cdots \eta_1^{a_r-\lambda_{n-1}-\cdots-\lambda_{n-(r-1)}-(r-1)} \\ &\quad \cdots \eta_{n-2}^{a_{n-2}}\eta_{n-3}^{a_{n-2}-\lambda_{n-1}-1} \cdots \eta_1^{a_{n-2}+3-\lambda_3-\cdots-\lambda_{n-1}-n} \\ &\quad \eta_{i-2}^{-\lambda_{i-1}-1}\eta_{i-3}^{-\lambda_{i-1}-\lambda_{i-2}-2} \cdots \eta_1^{-\lambda_3-\cdots-\lambda_{i-1}-(i-2)} \end{aligned} \quad (3.59)$$

for  $i = 1, \dots, n-1, n$ , where we treat

$$\eta_{i-2}^{-\lambda_{i-1}-1} \eta_{i-3}^{-\lambda_{i-1}-\lambda_{i-2}-2} \dots \eta_1^{-\lambda_i-\dots-\lambda_{i-1}-(i-2)} = 1 \quad \text{if } i = 1, 2. \quad (3.60)$$

Moreover, we set

$$\psi_n = 1, \quad \psi_i = \sigma_{n-1} \sigma_{n-2} \dots \sigma_i(1) = \eta_{n-1}^{n-i+\sum_{p=1}^{n-i} \lambda_{n-p}} \eta_{n-2}^{n-i-1+\sum_{p=2}^{n-i} \lambda_{n-p}} \dots \eta_i^{\lambda_i+1}(1), \quad (3.61)$$

for  $i = 1, 2, \dots, n-1$ . Then  $\{\psi_i \mid i = 1, \dots, n-1, n\}$  are solutions of (2.27) by Theorem 2.5. Denote

$$\lambda_{n-1,n} = 0, \quad \lambda_{n-1,i} = n-i + \sum_{p=1}^{n-i} \lambda_{n-p} \quad \text{for } i = 1, 2, \dots, n-1. \quad (3.62)$$

According to (3.50),

$$a_{n-1} = \lambda_{n-1,i_{n-1}} \quad \text{for some } i_{n-1} \in \{1, \dots, n-1, n\}. \quad (3.63)$$

Thus,

$$z = x_{2,1}^{a_1} \phi_{\bar{a}} = \Psi_{i_{n-1}} \psi_{i_{n-1}}. \quad (3.64)$$

Set

$$\lambda_j^{(n-2)} = \begin{cases} \lambda_j & \text{if } j < i_{n-1} - 1, \\ \lambda_{i_{n-1}} + \lambda_{i_{n-1}-1} + 1 & \text{if } j = i_{n-1} - 1, \\ \lambda_{j+1} & \text{if } i_{n-1} \leq j \leq n-2 \end{cases} \quad (3.65)$$

for  $j = 1, 2, \dots, n-2$ . By (2.29) and (2.30),

$$h_j(\psi_{i_{n-1}}) = \zeta_j(\psi_{i_{n-1}}) = \lambda_j^{(n-2)} \psi_{i_{n-1}} \quad \text{for } j = 1, 2, \dots, n-2. \quad (3.66)$$

Define

$$\begin{aligned} \bar{d}_i &= (\lambda_i^{(n-2)} - \sum_{j=i+1}^{n-1} x_{j,i} \partial_{j,i} + \sum_{j=i+2}^{n-1} x_{j,i+1} \partial_{j,i+1}) \partial_{i+1,i} \\ &+ \sum_{j=1}^{i-1} x_{i,j} \partial_{i+1,j} - \sum_{j=i+2}^{n-1} x_{j,i+1} \partial_{j,i} \end{aligned} \quad (3.67)$$

for  $i = 1, 2, \dots, n-2$  by (2.25). Then the system

$$d_i(\Psi_{i_{n-1}} \psi_{i_{n-1}}) = [d_i, \Psi_{i_{n-1}}](\psi_{i_{n-1}}) = 0 \quad \text{for } i = 1, 2, \dots, n-2 \quad (3.68)$$

is equivalent to the system

$$\bar{d}_i(\Psi_{i_{n-1}}(1)) = 0 \quad \text{for } i = 1, 2, \dots, n-2 \quad (3.68)$$

by Lemma 2.2, (3.64) and (3.66). The system (3.68) is a version (2.27) for  $sl(n-1)$ . By induction on  $n$ , there exist a  $\sigma' \in S_{n-1}$  and a constant  $c \in \mathbb{C}$  such that  $\Psi_{i_{n-1}} \psi_{i_{n-1}} =$

$c\sigma'(\psi_{i_{n-1}})$ . Thus  $z = c\sigma'(1)$  or  $z = c\sigma'\sigma_{n-1}\sigma_{n-1}\cdots\sigma_{i_{n-1}}(1)$  for some  $i_{n-1} \leq n-1$ . The last conclusion follows from (2.58) and induction.  $\square$

If

$$n-1 + \sum_{p=1}^{n-1} \lambda_p \in \mathbb{N} + 1, \quad (3.69)$$

then

$$\phi = \sigma_1 \cdots \sigma_{n-2} \sigma_{n-1} \sigma_{n-2} \cdots \sigma_1(1) \quad (3.70)$$

is a polynomial and so  $\tau^{-1}(\phi)$  (cf. (2.22)) is a nontrivial singular vector in the Verma module  $M_\lambda$  (cf. (2.14)), which was obtained by Malikov, Feigin and Fuchs [MFF]. In general, for  $1 \leq i < j \leq n-1$ ,

$$\phi_{i,j} = \sigma_i \cdots \sigma_{j-1} \sigma_j \sigma_{j-1} \cdots \sigma_i(1) \text{ is a polynomial if } j-i+1 + \sum_{r=i}^j \lambda_r \in \mathbb{N}. \quad (3.71)$$

By (3.50) and induction on  $n$ , we have:

**Corollary 3.2.** *The Verma module  $M_\lambda$  is irreducible if and only if*

$$j + \sum_{p=0}^{j-1} \lambda_{i+p} \notin \mathbb{N} \quad \text{for } 1 \leq i \leq n-1, 0 \leq j \leq n-i. \quad (3.72)$$

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